4 cópias – Vol. 4

Chapman

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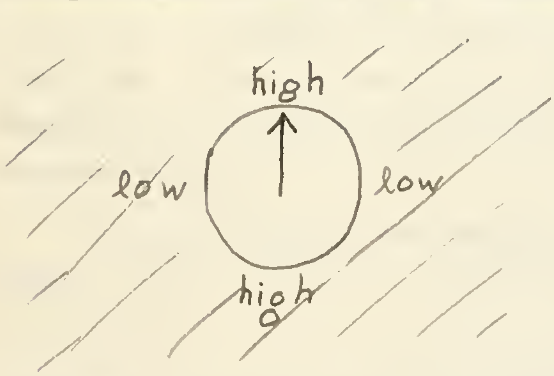
The Theory of a Pulsating Ellipsoidal Gas Bubble

I. Introduction

One of The major outstanding problems in the theory of the pulsating gas bubble produced by an underwater explosion is the deviation of the bubble from the spherical shape, and the effect of this deviation on the secondary pressure pulse. As a step to the solution of this problem, we shall suppose that the gas bubble is constrained to be ellipsoidal in shape. The flow of the water due to a pulsating, moving ellipsoid can be determined explicitly in terms of elementary functions and leads to a system of differential equations for the behavior of the bubble. These differential equations are derived in this note, and they show that the deviation from spherical shape is very marked when the bubble is in its contracted stage near the minimum size. More exact information awaits the numerical solution of these equations, which is under way.

The reason for a change in shape from the spherical is the following. The bubble in expanding and contracting imparts a linear momentum to the surrounding water, due to the buoyant action of gravity and the presence of nearby surfaces. This causes the bubble to move, and because of conservation of momentum, the motion is rapid when the bubble is near its minimum size.



It is a result of classical hydrodynamics that a moving sphere has a high pressure at the nose and rear of the sphere and a low pressure along the sides (see the diagram). Thus the gas bubble in moving will be flattened out in the direction of motion. This flattening process can be followed by assuming the bubble to be ellipsoidal. (Of course, it is quite possible for the change in shape to become more drastic than the ellipsoidal; e.g. the shape might become rather wavy or it might actually become toroidal.)

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II. The kinetic energy of the water.

1. To describe the ellipsoidal bubble, we shall use spheroidal coordinates. See Lamb, Hydrodynamics, 6th edition, Chap. V, esp. p.142-6. The bubble is assumed to be an oblate spheroid (planetary ellipsoid), with semi-axes α, α, γ where $\gamma < \alpha$. The intersection of the zx-plane with the spheroid is an ellipse with foci in the x-axis at a distance k from the origin, where $k = \sqrt{\alpha^2 - \gamma^2}$. See the diagram. Spheroidal coordinates are introduced in space. These are coordinates (ζ, μ, φ) where $0 \leq \zeta \leq \infty$, $-1 \leq \mu \leq 1$, $0 \leq \varphi \leq 2\pi$ and where x,y,z are related to ζ, μ, φ through the equations:

$$
(1) \qquad \begin{cases} x = k\sqrt{1+\zeta^2}\,\sqrt{1-\mu^2}\cos\varphi & 0 \leq \zeta \leq \infty \\ y = k\sqrt{1+\zeta^2}\,\sqrt{1-\mu^2}\sin\varphi & 0 \leq \mu \leq 1 \\ z = k\zeta\mu & 0 \leq \varphi \leq 2\pi \end{cases}
$$

where k is the semi-focal distance of the spheroidal bubble. The coordinates ζ, μ are more customarily presented as $\zeta = \sinh\eta$, $\mu = \cos\theta$; then $\sqrt{1+\zeta^2} = \cosh\eta$, $\sqrt{1-\mu^2} = \sin\theta$. The coordinate surfaces ζ = constant are a family of confocal ellipsoids, with semi-focal distance k and semi-axes $k\sqrt{1+\zeta^2}$, $k\sqrt{1+\zeta^2}$, $k\zeta$ respectively.

In particular, the equation of the surface of the spheroidal bubble is

$$\zeta = \zeta_0 \quad \text{(a constant).}$$

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Thus, the spheroidal bubble is characterized by two quantities namely k and ζ_0. (The semi-axes α, α, γ will no longer be used.) The coordinates (ζ, μ, φ) defined by (1) are then the spheroidal coordinates of an arbitrary point in space. The element of length dS in space expressed in shperoidal coordinates is

$$(2)\quad dS^2 = \frac{k^2(\zeta^2+\mu^2)}{1+\zeta^2}\, d\zeta^2 + \frac{k^2(\zeta^2+\mu^2)}{1-\mu^2}\, d\mu^2 + k^2(1+\zeta^2)(1-\mu^2)\, d\varphi^2$$

2. We shall now consider the flow produced in the water surrounding the ellipsoidal bubble when the ellipsoid changes its shape. The change in the ellipsoid can be produced by changing either k or ζ_0, and we shall first suppose that k changes, while ζ_0 is fixed. This corresponds to a similarity transformation, and is a "pure" expansion. If we follow a point on the surface of the ellipsoidal bubble with given coordinates (ζ_0, μ, φ), then the velocity of that point is by (1):

$$(3)\quad \begin{cases} \dot{x} = \dot{k}\sqrt{1+\zeta_0^2}\,\sqrt{1-\mu^2}\cos\varphi \\ \dot{y} = \dot{k}\sqrt{1+\zeta_0^2}\,\sqrt{1-\mu^2}\sin\varphi \\ \dot{z} = \dot{k}\,\zeta_0\,\mu \end{cases}$$

On the other hand, the direction of the normal to the ellipsoid at the point (ζ_0, μ, φ) is given, also from (1), by

$$(4)\quad \begin{cases} \dfrac{\partial x}{\partial \zeta} = k\,\dfrac{\zeta_0}{\sqrt{1+\zeta_0^2}}\,\sqrt{1-\mu^2}\cos\varphi \\ \dfrac{\partial y}{\partial \zeta} = k\,\dfrac{\ }{\sqrt{1+\zeta_0^2}}\,\sqrt{1-\mu^2}\sin\varphi \\ \dfrac{\partial z}{\partial \zeta} = k\mu \end{cases}$$

(direction normal to ellipsoid at point(ζ_0, μ, φ).)

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The component V_n of the velocity of the surface in the direction normal to the ellipsoid is

$$(5) \qquad V_n = \frac{\dot{x}\frac{\partial x}{\partial \zeta} + \dot{y}\frac{\partial y}{\partial \zeta} + \dot{z}\frac{\partial z}{\partial \zeta}}{\sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}} .$$

A simple calculation using (3) and (4) gives

$$(6) \qquad V_n = \dot{k}\,\zeta_0 \frac{\sqrt{1+\zeta_0^2}}{\sqrt{\zeta_0^2+\mu^2}} .$$

The boundary condition for the potential function Φ_1 describing the flow is

$$(7) \qquad -\frac{\partial \Phi_1}{\partial n} = V_n \quad \text{on } \zeta = \zeta_0$$

By (2),

$$(8) \qquad -\frac{\partial \Phi_1}{\partial n} = \frac{1}{\dfrac{k\sqrt{\zeta_0^2+\mu^2}}{\sqrt{1+\zeta_0^2}}} \left(-\frac{\partial \Phi_1}{\partial \zeta}\right) \quad \text{on } \zeta = \zeta_0$$

Combining (6), (7), (8), we therefore have

$$-\frac{\partial \Phi_1}{\partial \zeta} = \dot{k}\,k\,\zeta_0 \qquad \text{on } \zeta = \zeta_0$$

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Thus, $-\frac{\partial \Phi_1}{\partial \zeta}$ is constant on the surface of the ellipsoid, and the potential function Φ_1 is therefore easy to construct. We have in fact

$$\Phi_1 = \dot{k} \frac{k\zeta_o}{-q_o'(\zeta_o)} q_o(\zeta) \tag{9}$$

where $q_o(\zeta) = \cot^{-1}\zeta$, and the prime on $q_o'(\zeta_o)$ means the derivative with respect to ζ, (see Lamb, Chap. V, esp. page 143 for rotationally symmetric spheroidal harmonics).

These harmonics have the form

$$P_n(\mu) \quad q_n(\zeta)$$

where $P_n(\mu)$ are Legendre polynomials, and $q_n(\zeta)$ are, except for multiplying factors making them real, Legendre functions of the second kind for imaginary arguments. The first few of these functions are

$$\begin{aligned} P_o(\mu) &= 1 & q_o(\zeta) &= \cot^{-1}\zeta \\ P_1(\mu) &= \mu & q_1(\zeta) &= 1 - \zeta\cot^{-1}\zeta \\ P_2(\mu) &= \tfrac{1}{2}(3\mu^2-1) & q_2(\zeta) &= \tfrac{1}{2}(3\zeta^2+1)\cot^{-1}\zeta - \tfrac{3}{2}\zeta \end{aligned} \tag{10}$$

3. Consider the change in the ellipsoidal bubble due to varying ζ_o, keeping k fixed. Following a point on the surface of the ellipsoidal bubble with fixed μ, φ, we have for the velocity of that point, by (1):

$$\left\{ \begin{aligned} \dot{x} &= k \frac{\zeta_o \dot{\zeta}_o}{\sqrt{1+\zeta_o^2}} \sqrt{1-\mu^2}\cos\varphi \\ \dot{y} &= k \frac{\zeta_o \dot{\zeta}_o}{\sqrt{1+\zeta_o^2}} \sqrt{1-\mu^2}\sin\varphi \\ \dot{z} &= k \dot{\zeta}_o \mu \end{aligned} \right.$$

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The component V_n of the velocity of the surface in the direction normal to the ellipsoid is, by (4) and (5),

$$V_n = \dot{\zeta}_o k \sqrt{\frac{\zeta_o^2 + \mu^2}{1 + \zeta_o^2}}$$

The boundary condition for the potential function Φ_2 describing the flow is, by (7) and (8)

$$- \frac{\partial \Phi_2}{\partial \zeta} = \dot{\zeta}_o k^2 \frac{\zeta_o^2 + \mu^2}{\sqrt{1 + \zeta_o^2}} \qquad \text{or } \zeta = \zeta_o \quad .$$

This condition can be written in the following form:

$$- \frac{\partial \Phi_2}{\partial \zeta} = \dot{\zeta}_o \frac{k^2}{1 + \zeta_o^2} \left(\frac{2}{3} P_2(\mu) + \frac{3\zeta_o^2 + 1}{3} \right) \quad \text{or } \zeta = \zeta_o$$

where $P_2(\mu) = \dfrac{3\mu^2 - 1}{2}$. The solution for Φ_2 is

$$(11) \; \Phi_2 = \frac{\dot{\zeta}_o k^2}{3(1 + \zeta_o^2)} \left[\frac{2}{-q_2'(\zeta_o)} P_2(\mu) q_2(\zeta) + \frac{3\zeta_o^2 + 1}{1 \, q_o'(\zeta_o)} q_o(\zeta) \right]$$

4. Finally, consider the ellipsoidal bubble as fixed in size and shape, but moving in the z-direction with velocity $\dot{B}$. (The quantity B représents the vertical distance of the center of the bubble from same level.) For the velocity of a point on the surface of the bubble moving vertically upward, we have

$$\frac{\partial x}{\partial t} = 0, \quad \frac{\partial y}{\partial t} = 0, \quad \frac{\partial z}{\partial t} = \dot{B} \quad .$$

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The normal component of the velocity is, by (4) and (5),

$$v_n = \dot{B}\mu\sqrt{\frac{1+\zeta_o^2}{\mu^2+\zeta_o^2}} \; .$$

By (7) and (8), the boundary condition for the potential function Φ_3 is:

$$-\frac{\partial\Phi_3}{\partial\zeta} = \dot{B}k\mu \qquad \text{or} \quad \zeta = \zeta_o \; .$$

The solution for Φ_3 is

$$\Phi_3 = \frac{\dot{B}\,k}{-q_1'(\zeta_o)}\,\mu q_1(\zeta) \; . \tag{12}$$

5. The potential function Φ for the combined flow of the water, due to the changing shape of the ellipsoid and its motion through the water, is the sum of the individual parts Φ_1, Φ_2, Φ_3 given in (9), (11), (12). It is more simply expressed if the volume V of the bubble is introduced. The volume V is

$$V = \frac{4}{3}\pi\alpha^2\gamma = \frac{4}{3}\pi\, k^3\zeta_o(1+\zeta_o^2) \; .$$

The rate of change of colume $\dot{V}$ is simply expressed in terms of $\dot{k}$ and $\dot{\zeta}_o$.

The potential function $\Phi = \Phi_1 + \Phi_2 + \Phi_3$ for the combined flow is, from (9), (11), (12),

$$\Phi = \frac{\dot{V}}{4\pi k}\,q_o(\zeta) + \frac{\dot{B}\,k}{-q_1'(\zeta_o)}\,\mu q_1(\zeta) + \frac{\dot{\zeta}_o\,2k^2}{3(1+\zeta_o^2)(-q_2'(\zeta_o))}\,P_2(\mu)q_2(\zeta) \; . \tag{13}$$

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The convenient form $\frac{\dot{V}}{4\pi k} q_o(\zeta)$ for the first term is a matter for simple calculation, but it can also be obtained by the following general argument. At a large distance R from the center of the bubble, the potential function must by conservation of mass be of the form

$$\Phi \sim \frac{\dot{V}}{4\pi} \cdot \frac{1}{R} \quad .$$

At large distances from the origin

$$\frac{q_o(\zeta)}{k} \sim \frac{1}{k} \sim \frac{1}{R} \quad ,$$

whereas all the other terms in (13) vanish to a higher order in $\frac{1}{R}$.

6. The kinetic energy $\mathcal{T}$ of the water can now be obtained. We have by Green's formula,

$$\mathcal{T} = \frac{1}{2} \rho \iiint (\mathrm{grad}\, \Phi)^2 d\tau = \frac{1}{2} \rho \iint \Phi \left(- \frac{\partial \Phi}{\partial n}\right) dS \quad ,$$

where ρ is the density of the water and the integral is extended over the surface of the ellipsoid. Using (8) and

$$dS = k \sqrt{\frac{\zeta_o^2 + \mu^2}{1 - \mu^2}}\, d\mu \cdot k \sqrt{(1 + \zeta_o^2)(1 - \mu^2)}\, d\varphi$$

from (2), we find

$$\mathcal{T} = \frac{1}{2} \rho \iint k(1 + \zeta_o^2) \Phi \left(- \frac{\partial \Phi}{\partial \zeta}\right) d\mu \, d\varphi$$

(14)

$$= \pi \rho k (1 + \zeta_o^2) \int_{-1}^{1} \Phi \left(- \frac{\partial \Phi}{\partial \zeta}\right) d\mu$$

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From (13), we have

$$-\frac{\partial \Phi}{\partial \zeta}\Big]_{\zeta_o} = \frac{\dot{V}}{4\pi k(1+\zeta_o^2)} + \dot{B}k\mu + \frac{\dot{\zeta}_o \cdot 2k^2}{3(1+\zeta_o^2)} P_2(\mu) .$$

Substituting (13) and (15) into (16) and using the orthogonality of the Legendre polynomials $P_n(\mu)$, we find

(16) $$\mathcal{T} = 2\pi\rho \left[\frac{q_o(\zeta_o)}{16\pi^2 k} \dot{V}^2 + \frac{k^3(1+\zeta_o^2)}{3} \left(\frac{q_1(\zeta_o)}{-q_1'(\zeta_o)} \dot{B}^2 \quad \frac{4k^5}{45(1+\zeta_o^2)} \left(\frac{q_2(\zeta_o)}{-q_2'(\zeta_o)} \right) \dot{\zeta}_o^2 \right] .$$

The quantity k can be expressed in terms of V and ζ_o. Instead of V we shall use, however, the radius A of the sphere with the same volume. Thus, set

(17) $$V = \frac{4}{3}\pi A^3$$

The relation between k and V yields

$$k = \frac{A}{\zeta_o^{1/3}(1+\zeta_o^2)^{1/3}} .$$

These will be substituted in (17). But one final simplification the subscript o on ζ_o will henceforth be dropped. The result is

(18) $$\frac{\mathcal{T}}{2\pi} = m_o(\zeta) \cdot A^3\dot{A}^2 + m_1(\zeta) \cdot \frac{1}{6}A^3\dot{B}^2 + m_2(\zeta) \cdot \frac{1}{2}A^5\dot{\zeta}^2$$

where

(19) $$\begin{cases} m_o(\zeta) = \zeta^{1/3}(1+\zeta^2)^{1/3} q_o(\zeta) \\ m_1(\zeta) = \dfrac{2}{\zeta} \cdot \dfrac{q_1(\zeta)}{-q_1'(\zeta)} \\ m_2(\zeta) = \dfrac{8}{135} \dfrac{1}{\zeta^{2/3}(1+\zeta^2)^{8/3}} \cdot \dfrac{3}{\zeta} \cdot \dfrac{q_2(\zeta)}{-q_2'(\zeta)} . \end{cases}$$

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In the formulae, the size and shape of the ellipsoidal bubble are described by the following two quantities:

(1) The radius A of thesphere with the same volume as the ellipsoid.

(2) The quantity ζ which determines the shape of the ellipsoid and represents the ratio of the semi-minor axis γ to the semi-focal distance k,

$$\zeta = \frac{\gamma}{k} .$$

In terms of the semi-major and semi-minor axes and ,

$$\frac{\gamma}{\alpha} = \frac{\zeta}{\sqrt{1+\zeta^2}} .$$

The position of the ellipsoid is determined by the vertical distance B of its center from some horizontal level.

The quantity ζ can vary from 0 to ∞ , the case $\zeta = \infty$ corresponding to a sphere and the case $\zeta = 0$ to a flat circular disc. For the spherical case, $\zeta = \infty$. we have $m_o = 1$, $m_1 = 1$, $m_2 = 0$ and the kinetic energy formula (19) reduces to the known spherical case.

In view of (18), it is convenient to introduce a new variable λ in place of ζ . Set

$$(20) \qquad \lambda = \int_{\zeta}^{\infty} \sqrt{m_2(\zeta)}\,d\zeta .$$

As ζ varies from ∞ to 0, λ varies from 0 tothe finite value $\int_0^{\infty} \sqrt{m_2(\zeta)}\,d\zeta$, the value $\lambda = 0$ corresponding to a sphere.
In terms of λ the kinetic energy is

$$(21) \qquad \frac{T}{2\pi} = m_o \cdot A^3\dot{A}^2 + m_1 \cdot \frac{1}{6} A^3\dot{B}^2 + \frac{1}{2} A^5\dot{\lambda}^2$$

where m_o, m_1 are functions of λ defined through (19) and (20).

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III. The equations of motion of the bubble.

1. The potential energy U of the gas bubble is

$$U = \rho g(D_0 - B)\,\frac{4}{3}\pi A^3 + G(A)$$

where D_0 as the initial depth of the explosion from 33 ft. above sea level, B is the upward displacement of the bubble at any time, and G(A) is the internal energy of the gas. The term $\rho g(D_0 - B)$ represents the hydrostatic pressure at the center of the bubble.

The energy equation is

$$\mathcal{T} + U = E \text{ (a constant).}$$

Introducing dimensionless variables as in the AMG-NYU report "The best location of a mine near the bottom of the sea" by Friedman and Shiffman, the energy equation becomes

$$(22) \qquad \underbrace{m_0 \cdot a^3 \dot{a}^2 + m_1 \cdot \frac{1}{6} a^3 \dot{b}^2 + \frac{1}{2} a^5 \dot{\lambda}^2}_{\mathcal{T}} + \underbrace{\frac{d_0 - b}{d_0}\, a^3 + \frac{k}{a^{3/4}}}_{U} = 1$$

In addition to the Lagrange equation for b, there is a Lagrange equation for λ. These equations are

$$(23) \qquad \frac{d}{dt}\left(m_1 \cdot \frac{1}{3} a^3 \dot{b}\right) = \frac{1}{d_0}\, a^3$$

$$(24) \qquad \frac{d}{dt}\left(a^5 \dot{\lambda}\right) = \frac{dm_0}{d\lambda} \cdot a^3 \dot{a}^2 + \frac{dm_1}{d\lambda} \cdot \frac{1}{6} a^3 \dot{b}^2 .$$

The equation (22), (23), (24) are the differential equations for determining a, b, λ as functions of t.

2. The interest centers around the contracting stage, especially when the bubble is near its minimum size. At this time, equation (23) shows that the linear momentum

$$(25) \qquad m_1 \cdot \frac{1}{3} a^3 \dot{b} = \bar{s}$$

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is practically constant and equal to $\bar{s}$, which can be calculated in terms of d_o. In case there are nearby rigid walls or free surfaces, this conclusion is still valid, except that $\bar{s}$ now depends on the distances to the nearby objects. Substitution in (22) and (24) gives approximately

$$(26) \qquad m_o \cdot a^3 \dot{a}^2 + \frac{3\bar{s}^2}{2m_1 a^3} + \tfrac{1}{2} a^5 \dot{\lambda}^2 + a^3 + \frac{k}{a^{3/4}} = 1$$

$$(27) \qquad \frac{d}{dt}(a^5 \dot{\lambda}) = \frac{dm_o}{d\lambda} \cdot a^3 \dot{a}^2 + \frac{1}{m_1^2} \frac{dn_1}{d\lambda} \cdot \frac{3\bar{s}^2}{2a^3}$$

These equations are still approximately valid throughout the entire contracting state, since for larger a the errors made in using the equation (25) are small.

In the case where there are nearby surfaces, the only appreciable effect on the equations (26),(27), besides the influence on the value of the momentum constant $\bar{s}$, is an extra factor $(1+f_o)$ occurring in the terms $a^3\dot{a}^2$ in (26) and (27). All other effects are negligible.

3. An analysis of equation (27) shows the extreme instability of the shape of the bubble. Near the minimum size, the "shape" momentum

$$a^5 \dot{\lambda} = \Lambda$$

has a value given by the time integral of the right hand aide of (27). Then

$$\dot{\lambda} = \frac{\Lambda}{a^5}$$

which shows how large $\dot{\lambda}$ can become for small a. Thus λ may become large and near its upper limit, which means a flat ellipsoid. In this case m_o is small, and m_1 is large. Also both $\frac{dm_o}{d\lambda}$ and $\frac{1}{m_1^2}\frac{dm_1}{d\lambda}$ are small. Exact statements about the size of λ require the numerical integration of (26) and (27).

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4. The displacement of the bubble can be determined from (25) by solving for $\dot{b}$

$$\dot{b} = \frac{3\bar{s}}{m_1 a^3}$$

and integrating with respect to t. This shows that for large m_1, $\dot{b}$, Δb and the translational energy $\dfrac{3\bar{s}^2}{2m_1 a^3}$ of the water are small.

The pressure pulse P emitted by the contracting bubble is determined from Bernoulli's equation

$$P = \rho \frac{\partial \Phi}{\partial t}$$

where the remaining terms are dropped because they ~~are~~ die down at a faster rate than 1/R. Using the first term of (13), which is dominant for large distances, we have in dimensionless form

$$p = (a^2 \dot{a})^{\cdot}$$

Differentiating (26) with respect to t, we see that the peak pressure $\bar{p}$ is expressed with the coefficient m_o in the denominator. Thus, the smaller m_o is, the greater as the peak pressure $\bar{p}$ of the secondary pulse.

All of the remarks in this section apply to any shape bubble.